JOURNAL OF COMBINATORIAL THEORY, Series A 63, 275–300 (1993)

Partition Identities and a Continued Fraction of Ramanujan

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Communicated by George Andrews

Received February 10, 1991

We study the numerator and denominator of a continued fraction $R(a, b)$ of Ramanujan and establish the equality of various restricted partition functions. We use the continued fraction to give a unified approach to several partition identities some of which generalize results of Bressoud and Göllnitz. We also give a combinatorial interpretation for the coefficients in the power series expansion of the reciprocal $1/R(-a, -b)$, extending a result of Odlyzko and Wilf. © 1993 Academic Press, Inc.

1. INTRODUCTION

In this paper we study a continued fraction of Ramanujan and establish the equality of various restricted partition functions. The fraction in question was mentioned by Ramanujan in his second letter to Hardy [11, p. xxviii]; it is

$$R(a, b) = 1 + \cfrac{bq}{1 + aq + \cfrac{bq^2}{1 + aq^2 + \cfrac{bq^3}{\cdots}}}. \tag{1.1}$$

Note that $R(0, 1)$ is the celebrated Rogers–Ramanujan continued fraction, which has been investigated in great detail (see Andrews [3, Chap. 7], for example). In the course of analyzing identities from Ramanujan's Lost Notebook [13], Andrews [4, 5] has discussed the fraction $R(a, b)$, but

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mainly from the viewpoint of transformation formulas. Our emphasis here is on using (1.1) to give a unified treatment of several partition identities, some of which are new.

In Section 2, we analyze the numerator and denominator of $R(a, b)$, and obtain various partition identities leading to generalizations of some theorems of Bressoud [7] and Göllnitz [8], of which the following is typical. For convenience we adopt the convention that $p \equiv \alpha \pmod{m}$ means $p = \alpha + (\lambda - 1)m$ with $\lambda \geqslant 1$; we say that p has *level* λ. The modulus m is usually fixed in a given discussion. If $\pi: n = n_1 + n_2 + \cdots + n_k$ is a partition of n, a part $n_i = \alpha \pmod{m}$ is called an α-part of π. We are now in position to state:

THEOREM A. *Suppose $m \geqslant 2$, and α, β are positive integers with $\alpha < \beta < \alpha + m$.*

Let $A(n; i, j)$ be the number of partitions of n into $i + j$ distinct α-parts and j distinct $(\beta - \alpha)$-parts such that the level of each $(\beta - \alpha)$-part is $\leqslant i + j$.

Let $B(n; i, j)$ be the number of partitions of n into i distinct α-parts and j distinct β-parts such that each α-part is $\geqslant mj + \alpha$ and no two β-parts have consecutive levels.

Let $C(n; i, j)$ be the number of partitions of n into i α-parts and j β-parts with difference $\geqslant m$ between parts and such that no two β-parts have consecutive levels.

Then $A(n, i, j) = B(n; i, j) = C(n; i, j)$.

In the special case where $\alpha = m$ and $\beta < 2m$, Theorem A reduces to a result of Bressoud [7, Corollary 2] (which is itself an extension of a theorem of Göllnitz [8, Satz 3.1]), after an elementary transformation of the conditions defining $B(n; i, j)$; see (2.7) and (2.8) below. Neither Bressoud nor Göllnitz consider the function $A(n; i, j)$, whose role will become clear in Section 3, where by choosing the values of a, b and q suitably in (1.1) we give a unified combinatorial treatment of several partition identities, some of which are new. Examples include Lebesgue's identity

$$\sum_{k \geqslant 0} \frac{q^{k(k+1)/2} \prod_{j=1}^{k} (1 + bq^j)}{(1-q)(1-q^2) \cdots (1-q^k)} = \prod_{m \geqslant 1} (1 + bq^{2m})(1 + q^m) \tag{1.2}$$

and Sylvester's refinement of Euler's theorem on partitions into odd parts and distinct parts.

In Section 4 we give a combinatorial interpretation of the coefficients in the power series expansion of $1/R(-a, -b)$. This extends results of Odlyzko and Wilf [10] on the coefficients of the Rogers–Ramanujan reciprocal $1/R(0, -b)$.

Finally, in Section 5 we discuss special properties of a certain continued fraction obtained from $R(a, b)$ by a suitable choice of a, b and q.

We use the standard notation $(a)_n = (a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ and $(a)_\infty = \lim_{n \to \infty} (a)_n$.

2. Analysis of Ramanujan's Continued Fraction

For a continued fraction F, let P_n/Q_n denote its nth convergent, and suppose that $\lim_{n \to \infty} P_n = P$, $\lim_{n \to \infty} Q_n = Q$ in a suitable topology. We then say that F has numerator P and denominator Q, and write $P = F^{\mathrm{N}}$, $Q = F^{\mathrm{D}}$.

Consider the fraction

$$F(a, b) = 1 + a + \cfrac{acq}{1 + aq + \cfrac{acq^2}{1 + aq^2 + \cfrac{acq^3}{\cdots}}}.$$

This can be written in the form

$$F(a, c) = \frac{f(a, c)}{f(aq, c)},$$

where

$$f(a, c) = \sum_{k \geqslant 0} A_k a^k.$$

We now compute the coefficients $A_k = A_k(c, q)$. For this, observe that $f(a, c)$ satisfies the recurrence

$$f(a, c) = (1 + a) f(aq, c) + acqf(aq^2, c).$$

Therefore the coefficients A_k satisfy

$$A_k = q^k A_k + q^{k-1} A_{k-1} + cq^{2k-1} A_{k-1},$$

or equivalently

$$A_k = \frac{q^{k-1}(1 + cq^k)}{(1 - q^k)} A_{k-1}.$$

By iteration this yields

$$f(a, c) = \sum_{k \geqslant 0} \frac{a^k q^{k(k-1)/2} (-cq)_k}{(q)_k}.$$

Let $c=a^{-1}b$; then

$$R(a, b)=\frac{f(a, a^{-1}b)}{f(aq, a^{-1}b)}-a$$

is Ramanujan's fraction (1.1).

LEMMA 1. *For the fraction $R(a, b)$, the numerator is*

$$R^{\mathrm{N}}(a, b)=\sum_{k\geqslant 0}\frac{a^k q^{k(k+1)/2}(-a^{-1}b)_k}{(q)_k} \tag{2.1}$$

and the denominator is

$$R^{\mathrm{D}}(a, b)=\sum_{k\geqslant 0}\frac{a^k q^{k(k+1)/2}(-a^{-1}bq)_k}{(q)_k}. \tag{2.2}$$

Proof. The expansion (2.2) is an immediate consequence of

$$R^{\mathrm{D}}(a, b)=f(aq, a^{-1}b).$$

The expansion (2.1) is more complicated. To obtain it, observe that

$$\begin{aligned}
R^{\mathrm{N}}(a, b)&=f(a, a^{-1}b)-af(aq, a^{-1}b)\\
&=\sum_{k\geqslant 0}\frac{a^k q^{k(k-1)/2}(-a^{-1}bq)_k}{(q)_k}-\sum_{k\geqslant 0}\frac{a^{k+1}q^{k(k+1)/2}(-a^{-1}bq)_k}{(q)_k}\\
&=1+\sum_{k\geqslant 0}\frac{a^{k+1}q^{k(k+1)/2}(-a^{-1}bq)_k}{(q)_k}\left(\frac{1+a^{-1}bq^{k+1}}{1-q^{k+1}}-1\right)\\
&=1+\sum_{k\geqslant 0}\frac{a^{k+1}q^{(k+1)(k+2)/2}(-a^{-1}bq)_k\,(1+a^{-1}b)}{(q)_{k+1}}\\
&=\sum_{k\geqslant 0}\frac{a^k q^{k(k+1)/2}(-a^{-1}b)_k}{(q)_k},
\end{aligned}$$

as claimed.

Andrews [4, 5] considered the expansions in Lemma 1 while discussing a transformation formula of Ramanujan [13] for $R(a, b)$. Our emphasis here is on the partition theorems that can be derived using $R(a, b)$, and for this the following lemma is crucial.

LEMMA 2. *For the fraction $R(a, b)$ we also have the expansions*

$$R^{\mathrm{N}}(a, b)=\sum_{i,j\geqslant 0}\frac{a^i b^j q^{(i^2+i)/2+ij+j^2}}{(q)_i\,(q)_j} \tag{2.3}$$

and

$$R^{\mathrm{D}}(a, b)=\sum_{i, j \geqslant 0} \frac{a^i b^j q^{(i^2+i)/2+ij+j^2+j}}{(q)_i\,(q)_j}. \tag{2.4}$$

Proof. To obtain (2.3) and (2.4) from (2.1) and (2.2), we use the q-binomial theorem,

$$(-z)_k=\sum_{j=0}^{k} z^j q^{j(j-1)/2} \begin{bmatrix} k \\ j \end{bmatrix}$$

with $z=a^{-1}b$ and $z=a^{-1}bq$. Therefore

$$\begin{aligned} R^{\mathrm{N}}(a, b) &= \sum_{k \geqslant 0} \frac{a^k q^{k(k+1)/2}}{(q)_k} \sum_{j=0}^{k} \frac{a^{-j} b^j q^{j(j-1)/2} (q)_k}{(q)_j\,(q)_{k-j}} \\ &= \sum_{i, j \geqslant 0} \frac{a^i b^j q^{(i+j)(i+j+1)/2+j(j-1)/2}}{(q)_i\,(q)_j}, \end{aligned}$$

where $i=k-j$; this is equivalent to (2.3). To obtain (2.4), observe that

$$R^{\mathrm{D}}(a, b)=R^{\mathrm{N}}(a, bq)$$

by comparing (2.1) and (2.2).

THEOREM 1^{N}. *Let $A^{\mathrm{N}}(n; i, j)$ be the number of partitions of n into $i+j$ distinct red parts and j distinct blue parts such that one of the blue parts may be zero and every blue part is $\leqslant i+j-1$.*

Let $B^{\mathrm{N}}(n; i, j)$ be the number of partitions of n into i distinct red parts and j distinct non-consecutive blue parts such that every red part is $>j$.

Let $C^{\mathrm{N}}(n; i, j)$ be the number of partitions of n into i red parts and j blue parts such that all parts are distinct and after each blue part there is a gap of at least 2. Then

$$A^{\mathrm{N}}(n; i, j)=B^{\mathrm{N}}(n; i, j)=C^{\mathrm{N}}(n; i, j).$$

The above theorem relates to the numerator of the fraction (1.1). It has a companion corresponding to the denominator, which is

THEOREM 1^{D}. *Let $A^{\mathrm{D}}(n; i, j)$ be as in $A^{\mathrm{N}}(n; i, j)$ except that every blue part is >0 and $\leqslant i+j$.*

Let $B^{\mathrm{D}}(n; i, j)$ be as in $B^{\mathrm{N}}(n; i, j)$ except that part 1 cannot be blue.

Let $C^{\mathrm{D}}(n; i, j)$ be as in $C^{\mathrm{N}}(n; i, j)$ except that part 1 cannot be blue. Then

$$A^{\mathrm{D}}(n; i, j)=B^{\mathrm{D}}(n; i, j)=C^{\mathrm{D}}(n; i, j).$$

Proof of Theorem 1^{N}. We rewrite the summand in (2.1) as

$$\frac{q^{k(k+1)/2}}{(q)_k}\prod_{j=1}^{k}(a+bq^{j-1}). \tag{2.5}$$

We think of $q^{k(k+1)/2}/(q)_k$ as the generating function for the distinct red parts. The distinct blue parts are generated by the terms in the product involving b. Thus the interpretation for $A^{\mathrm{N}}(n; i, j)$ follows because $k=i+j$.

Next we consider the function $C^{\mathrm{N}}(n; i, j)$. In this case the term (2.5) is to be interpreted as the generating function for bipartitions $\pi=(\pi_1, \pi_2)$, where π_1 is a partition into k distinct red parts and π_2 is a partition into distinct blue parts, each $\leqslant k-1$. Consider the graph of the bipartition (π_1, π_2^*), where π_2^* is the Ferrers conjugate of the non-zero parts of π_2 and if π_2 has 0 as a part then the zero in π_2^* is the empty node at the intersection of column 1 and row $i+j$. Next, circle the nodes at the bottom of each column of π_2^* and put an empty circle corresponding to the part 0 (if 0 is a part). Now add the corresponding rows of π_1 and π_2^* to get an ordinary partition π_3. In π_3 the parts ending in circles are blue and the rest are red; if π_2^* has an empty circle, then this corresponds to the case where the smallest part of π_3 is blue. This construction yields the partitions counted by $C^{\mathrm{N}}(n; i, j)$, and is clearly reversible.

Finally, to get the interpretation for $B^{\mathrm{N}}(n; i, j)$, we rewrite (2.3) as follows:

$$\begin{aligned} R^{\mathrm{N}}(a, b) &= \sum_{j\geqslant 0}\frac{b^j q^{j^2}}{(q)_j}\sum_{i\geqslant 0}\frac{(aq^j)^i\, q^{i(i+1)/2}}{(q)_i} \\ &= \sum_{j\geqslant 0}\frac{b^j q^{j^2}}{(q)_j}(-aq^{j+1})_\infty. \end{aligned} \tag{2.6}$$

In (2.6) the term $b^j q^{j^2}/(q)_j$ is the generating function for partitions into j blue parts with minimal difference 2, and $(-aq^{j+1})_\infty$ generates partitions into distinct red parts, each $>j$. This proves Theorem 1^{N}, and the proof of Theorem 1^{D} is similar.

Theorem A of the introduction can be obtained from Theorem 1^{N} by "dilatation and translation." More precisely, replace q by q^m, a by $aq^{\alpha-m}$, and b by $abq^{\beta-m}$ in (2.1) and (2.3). The red parts thus become the α-parts and the blue parts become the $(\beta-\alpha)$-parts for the partition function A and the β-parts for the functions B and C. Theorem A can also be obtained from Theorem 1^{D}. In this case replace q by q^m, a by $aq^{\alpha-m}$, and b by $abq^{\beta-2m}$ in (2.2) and (2.4).

Remarks. (i) The assumption that $\beta<\alpha+m$ was made in Theorem A only for convenience. If $\beta=\alpha+m$, then we have to distinguish these parts with colors.

(ii) As pointed out earlier, Bressoud's Corollary 2 of [7] corresponds to the case $\alpha = m < \beta < 2m$. Instead of the partition function $B(n; i, j)$, Bressoud considers $B^*(n; i, j) =$ the number of partitions of n into j distinct β-parts (mod $2m$) and distinct m-parts (mod m) of which exactly i are $> jm$.

It turns out that

$$B(n; i, j) = B^*(n; i, j). \tag{2.7}$$

To demonstrate this equality we show that the generating functions of B and B^* are equal. When $\alpha = m$,

$$\begin{aligned}\sum_{n,i,j \geqslant 0} B(n; i, j)\, a^i b^j q^n &= \sum_{i,j \geqslant 0} \frac{a^i b^j q^{m((i^2+i)/2 + ij + j^2) + (\beta - m)j}}{(q^m; q^m)_i\, (q^m; q^m)_j} \\ &= \sum_{j \geqslant 0} \frac{b^j q^{m(j^2 - j) + \beta j}}{(q^m; q^m)_j} \sum_{i \geqslant 0} \frac{a^i q^{mij + m(i^2+i)/2}}{(q^m; q^m)_i} \\ &= \sum_{j \geqslant 0} \frac{b^j q^{m(j^2 - j) + \beta j}}{(q^m; q^m)_j} (-aq^{m(j+1)}; q^m)_\infty \\ &= \sum_{j \geqslant 0} \frac{b^j q^{m(j^2 - j) + \beta j}}{(q^{2m}; q^{2m})_j} (-q^m; q^m)_j\, (-aq^{m(j+1)}; q^m)_\infty .\end{aligned} \tag{2.8}$$

In (2.8) the factor

$$\frac{q^{m(j^2 - j) + \beta j}}{(q^{2m}; q^{2m})_j}$$

is the generating function for partitions into j distinct β-parts (mod $2m$), and the coefficient of a^i in

$$(-q^m; q^m)_j\, (-aq^{m(j+1)}; q^m)_\infty$$

is the generating function for partitions into distinct m-parts (mod m) of which exactly i are $> jm$. Thus (2.8) is the generating function of $B^*(n; i, j)$, and this proves (2.7).

(iii) We point out that the transition from B to B^* is possible only because $\alpha = m$, as is clearly seen in (2.8). In replacing B^* by B we have a partition function with modified conditions on the parts, and it is this which enables us to treat the case $\alpha \neq m$ in Theorem A. In fact if we take $a = 1$ in (2.8) a further simplification arises, because the expression in (2.8) has a product representation. This phenomenon is best understood without any dilatation, and indeed gives a proof of Lebesgue's identity (1.2) without recourse to Heine's transformation. To be precise, with $a=1$ we have from (2.2) and (2.4)

$$\begin{aligned}\sum_{k\geqslant 0}\frac{q^{k(k+1)/2}(-bq)_k}{(q)_k}&=\sum_{j\geqslant 0}\frac{b^j q^{j^2+j}}{(q)_j}\sum_{i\geqslant 0}\frac{q^{ij+(i^2+i)/2}}{(q)_i}\\&=\sum_{j\geqslant 0}\frac{b^j q^{j^2+j}(-q^{j+1})_\infty}{(q)_j}\\&=(-q)_\infty\sum_{j\geqslant 0}\frac{b^j q^{j^2+j}}{(q^2;q^2)_j}\\&=(-q)_\infty\,(-bq^2;q^2)_\infty,\end{aligned}$$

which is (1.2).

3. Further Combinatorial Interpretations

It is possible to give purely combinatorial proofs of all the partition identities and theorems in Section 2. Many of the essential ideas are contained in Bressoud [7], so we will not repeat them here. Lebesgue's identity (1.2), however, is not covered by Bressoud, and since it involves a genuine variation we discuss it in detail.

Combinatorial Proof of Lebesgue's Identity. For the series in (1.2), the coefficient of b^j is the generating function for bipartitions,

$$\pi=(\pi_1;\pi_2)=(a_1,a_2,...,a_s;b_1,b_2,...,b_j),$$

where the parts satisfy the conditions

$$\left.\begin{array}{c}a_1>a_2>\cdots>a_s\geqslant 1\\ b_1>b_2>\cdots>b_k\geqslant 1\\ s\geqslant b_1\end{array}\right\}. \tag{3.1}$$

Given a bipartition π, consider the Ferrers graph of π_1 and the graph of π_2^*, the Ferrers conjugate of π_2, as shown below:

$$\begin{array}{ccccccccccc} & & & \pi_1 & & & & & \pi_2^* & \\ a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot\\ a_3 & \cdot & \cdot & \cdot & & & & & \cdot & \cdot & \\ a_4 & \cdot & & & & & & & \cdot & & \\ & & & & & & & & b_1 & b_2 & b_3\end{array}$$

Note that the columns of π_2^* can be embedded into unique positions in π_1. This gives a partition π_3 whose Ferrers graph is shown below. Here the positions occupied by the embedded columns of π_2^* are indicated by vertical lines joining the dots:

π_3

Equivalently, one may think of π_3 as being obtained by adding the corresponding rows of π_1 and π_2^*.

Let $c_1, c_2, ..., c_s$ denote the parts of π_3. The embedding of π_2^* into π_1 has created j gaps $\geqslant 2$ between the parts $c_1, c_2, ..., c_s$, and 0. Whereas π_1 and π_2^* give rise to a unique π_3, in the opposite direction the process is not unique. For if the partition π_3 has k gaps $\geqslant 2$ between the parts $c_1, c_2, ..., c_s$ and 0 with $k \geqslant j$, then there are $\binom{k}{j}$ ways of extracting π_2^* from π_3. Thus if

$V(n; j) =$ the number of bipartitions $\pi = (\pi_1, \pi_2)$ where π_2 has exactly j parts

and

$G(n; k) =$ the number of partitions of n into distinct parts $c_1, ..., c_s$ with exactly k gaps between the parts $c_1, ..., c_s, 0$,

then the functions V and G are related as follows:

$$V(n; j) = \sum_{k \geqslant j} \binom{k}{j} G(n; k). \tag{3.2}$$

In terms of generating functions, this is

$$\begin{aligned}\sum_j V(n; j)\, b^j &= \sum_j b^j \sum_{k \geqslant j} \binom{k}{j} G(n; k)\\ &= \sum_k G(n; k) \sum_{j \leqslant k} \binom{k}{j} b^j\\ &= \sum_k (1+b)^k G(n; k). \qquad (3.3)\end{aligned}$$

In particular with $b=1$ in (3.3) we get

$$V(n)=\sum_k 2^k G(n;k), \tag{3.4}$$

where $V(n)=\sum_j V(n;j)$.

The partition functions V and G are combinatorial entities related to the left side of (1.2). To pass to the right side we follow Bressoud [7]. The coefficient of b^j on the right side of (1.2) is the generating function for bipartitions,

$$\pi^{(2)}=(\pi_1^{(2)};\pi_2),$$

where $\pi_1^{(2)}=2b_1+2b_2+\cdots+2b_j$ is a partition into j distinct even parts and $\pi_2=a_1+a_2+\cdots+a_l$ is a partition into distinct parts. We consider certain constructions below which will be illustrated for the case

$$\pi_1^{(2)}=10+8+6+2 \qquad \text{and} \qquad \pi_2=8+7+4+2+1.$$

Step 1. Decompose π_2 into partitions π_4 and π_5, where π_4 consists of the parts of π_2 which are $\leqslant j$, and π_5 has the remaining parts:

$$\pi_4=4+2+1, \qquad \pi_5=8+7.$$

Step 2. Embed the Ferrers conjugate π_4^* of π_4 into $\pi_1^{(2)}$. This yields a partition $\pi_6=d_1+d_2+\cdots+d_j$, where each gap between the parts $d_1, d_2, ..., d_j$ and 0 is $\geqslant 2$ and the number of odd gaps is the number of parts of π_4:

$$\pi_6=\pi_1^{(2)}+\pi_4^*=13+10+7+3.$$

Step 3. Write the parts of π_5 in a column in decreasing order, and below them write the parts of π_6 in decreasing order. Draw a horizontal line to separate the parts of π_5 and π_6.

Step 4. Subtract 0 from the bottom element, 1 from the next element above, 2 from the one above that, etc., and display the new values as well as the subtracted values in two adjacent columns $C_1 \mid C_2$.

Step 5. Rearrange the elements of C_1 in decreasing order to form a column C_1^{R}. (Note that C_1 and C_1^{R} each have at least j different parts.)

Step 6. Finally, add corresponding elements of C_1^{R} and C_2 to get a partition π_3 into distinct parts $c_1, c_2, ..., c_s$ with k gaps $\geqslant 2$ between $c_1, c_2, ..., c_s$ and 0, where $k \geqslant j$:

Step 3	Step 4	Step 5	Step 6
π_5/π_6	$C_1 \mid C_2$	$C_1^{\mathrm{R}} \mid C_2^*$	π_3
8	3 | 5	10 | 5	15
7	3 | 4	8 | 4	12
13	10 | 3	6 | 3	9
10	8 | 2	3 | 2	5
7	6 | 1	3 | 1	4
3	3 | 0	3 | 0	3

Thus starting with a bipartition $\pi^{(2)}$ given by the right side of (1.2), these steps lead to a partition π_3 of the type counted by G; this is on the left side of (1.2). Each of the steps is a one-to-one correspondence except Step 5. If we reverse these operations, then in going from Step 5 to Step 4 we can choose any j out of the k distinct elements of C_1^{R} and put them below the horizontal line in C_1. So if $E(n, j)$ denotes the number of bipartitions $\pi^{(2)} = (\pi_1^{(2)}; \pi_2)$ of n, where $\pi_1^{(2)}$ has j distinct parts, then we have the relation

$$E(n; j) = \sum_{k \geqslant j} \binom{k}{j} G(n; k). \tag{3.5}$$

Upon comparing (3.5) with (3.2) we see that

$$E(n; j) = V(n; j), \tag{3.6}$$

which completes the combinatorial proof of (1.2).

Remarks. (i) A more natural interpretation of $E(n; j)$ is

$E(n; j) =$ the number of partitions of n where even parts do not repeat and there are j even parts,

because

$$\sum_{n,j} E(n; j)\, b^j q^n = \prod_{m=1}^{\infty} (1 + bq^{2m})(1 + q^m) = \prod_{m=1}^{\infty} \frac{(1 + bq^{2m})}{(1 - q^{2m-1})}.$$

With this interpretation (3.5) was established by Andrews [2].

(ii) In this proof of Lebesgue's identity the function E corresponds to B in Theorems 1 and A, and B^* in (2.7). Similarly the bipartition function V corresponds to the function A, and G corresponds to C. Thus in the case of Lebesgue's identity the analogue of Bressoud's result is a relation between E and G. The more natural relation, however, is (3.6), and so (3.5) is analogous to (3.2). This demonstrates the role of V or more generally of the function A in such one-to-one correspondences.

(iii) If $E(n)=\sum_j E(n;j)$ is the number of partitions of n where even parts do not repeat, then (3.2), (3.4) and (3.5) imply

THEOREM 3. $E(n)=\sum_{k\geqslant 0} 2^k G(n;k)$.

This can be interpreted combinatorially as follows: let $n=c_1+c_2+\cdots+c_s$ be a partition into distinct parts with exactly k gaps $\geqslant 2$ between the parts $c_1, c_2, ..., c_s, 0$. Each gap $\geqslant 2$ is counted with weight 2 because we have a choice of extracting a part of π_2 out of it or not. Thus $G(n;k)$ has a weight 2^k attached to it. There must always be at least one gap $\geqslant 2$ between $c_1, c_2, ..., c_s$ and 0 unless $c_1=s$, $c_2=s-1, ..., c_s=1$. Therefore we have

COROLLARY 1. *$E(n)$ is odd if and only if n is* triangular.

Actually (3.2) and (3.5) imply that

$$\prod_{m=1}^{\infty}\frac{(1+bq^{2m})}{(1-q^{2m-1})}=\sum_{n,k}(1+b)^k\,G(n;k)\,q^n,$$

which when combined with Corollary 1 yields Gauss's identity

$$\prod_{m=1}^{\infty}\frac{(1-q^{2m})}{(1-q^{2m-1})}=\sum_{n\geqslant 0} q^{n(n+1)/2}$$

when $b=-1$. (Conversely, Corollary 1 is an easy consequence of Gauss's identity.)

(iv) Lebesgue's identity (1.2) shows that Ramanujan's fraction $R(a, b)$ has a product representation when $a=1$. More precisely (2.1) and (2.2) yield

$$1+\cfrac{bq}{1+q+\cfrac{bq^2}{1+q^2+\cfrac{bq^3}{\cdots}}}=\prod_{m=1}^{\infty}\frac{(1+bq^{2m-1})}{(1+bq^{2m})}. \tag{3.7}$$

The most interesting case of (3.7) is obtained with $q\mapsto q^2$ and $b\mapsto bq^{-1}$. This gives

$$1+\cfrac{bq}{1+q^2+\cfrac{bq^3}{1+q^4+\cfrac{bq^5}{}}}=\prod_{m=1}^{\infty}\frac{(1+bq^{4m-3})}{(1+bq^{4m-1})}. \tag{3.8}$$

From (1.2) and (2.1) it follows that the numerator of (3.8) is

$$\sum_{k\geq 0} \frac{q^{k(k+1)}(-bq^{-1};q^2)_k}{(q^2;q^2)_k} = \prod_{m=1}^{\infty} (1+bq^{4m-3})(1+q^{2m})$$
$$= \prod_{m=1}^{\infty} (1+bq^{4m-3})(1+q^{4m-2})(1+q^{4m}). \quad (3.9)$$

Similarly, for the denominator we have

$$\sum_{k\geq 0} \frac{q^{k(k+1)}(-bq;q^2)_k}{(q^2;q^2)_k} = \prod_{m=1}^{\infty} (1+bq^{4m-1})(1+q^{4m-2})(1+q^{4m}). \quad (3.10)$$

These identities can be interpreted combinatorially; we state only the partition theorem given by (3.10), which is a corollary of Theorem A.

COROLLARY 2^A. *Let $A(n;k)$ be the number of partitions of n into distinct parts, of which exactly k are odd, where each odd part is less than twice the number of even parts.*

Let $B^*(n;k)$ be the number of partitions of n into distinct parts $\equiv 2, 3, 4$ (mod 4), *of which k are* $\equiv 3$ (mod 4).

Let $C(n;k)$ be the number of partitions of n into parts each ≥ 2, *with minimal difference* 2, *of which k are odd and these are non consecutive.*

Then $A(n;k) = B^*(n;k) = C(n;k)$.

The partition functions $A(n;k)$ and $C(n;k)$ relate to the left side of (3.10), while $B^*(n;k)$ relates to the right side. By taking $m=2$ in Theorem A, we see that the correspondence between the functions A and C are actually stronger than the one in Corollary 2^A; it involves two parameters i and j. More precisely, let

$A(n;i,j) =$ the number of partition of n into $i+j$ distinct even parts and j distinct odd parts, each $<2(i+j)$,

and

$C(n;i,j) =$ the number of partitions of n into non-consecutive parts each >1, of which i are even, j are odd, with difference ≥ 4 between odd parts.

Then

$$A(n;i,j) = C(n;i,j). \quad (3.11)$$

Just as partition function A has a companion C, a companion F^* can be constructed for B^* using Cauchy's identity

$$\sum_{k \geqslant 0} \frac{(a)_k\, t^k}{(q)_k} = \frac{(at)_\infty}{(t)_\infty}, \tag{3.12}$$

and as in (3.11), here too there is a strong correspondence involving two parameters. More precisely, replace q by q^4, t by tq^2 and a by $-bt^{-1}q$ in (3.12) to get

$$\sum_{k \geqslant 0} \frac{(-bt^{-1}q; q^4)_k\, q^{2k} t^k}{(q^4; q^4)_k} = \prod_{m=0}^{\infty} \frac{(1+bq^{4m+3})}{(1-tq^{4m+2})}. \tag{3.13}$$

When $t=1$ the right side of (3.13) coincides with the right side of (3.10).

Let

$B_2^*(n; i, j) =$ the number of partitions of n into j distinct parts $\equiv 3 \pmod 4$ and i parts $\equiv 2 \pmod 4$.

Then

$$\prod_{m=0}^{\infty} \frac{(1+bq^{4m+3})}{(1-tq^{4m+2})} = \sum_{n,i,j} B_2^*(n; i, j)\, t^i b^j q^n. \tag{3.14}$$

The function B^* in (3.14) has a subscript 2 to distinguish it from the function B^* in (2.7). The two are of course related (see (3.18) below). With regard to the left side of (3.13), note that

$$\frac{t^k q^{2k}}{(q^4; q^4)_k}$$

is the generating function for partitions into k parts $\equiv 2 \pmod 4$, and the coefficient of a^j in $(-aq; q^4)_k$ is the generating function for partitions into j distinct parts $\equiv 1 \pmod 4$, with all parts $<4k$. Thus if

$F^*(n; i, j) =$ the number of partitions of n into $i+j$ parts $\equiv 2 \pmod 4$ and j distinct parts $\equiv 1 \pmod 4$, each $<4(i+j)$,

then

$$\sum_{k \geqslant 0} \frac{(-bt^{-1}q; q)_k\, q^{2k} t^k}{(q^4; q^4)_k} = \sum_{n,i,j} F^*(n; i, j)\, t^i b^j q^n. \tag{3.15}$$

Hence (3.13), (3.14), and (3.15) show that

$$F^*(n; i, j) = B_2^*(n; i, j), \tag{3.16}$$

which is analogous to (3.11).

Thus we have partition functions A, C, B_2^* and F^* with pairs of strong correspondences given by (3.11) and (3.16). But in order to get equality between all four functions we need to set $t=1$ and do away with the parameter i as in Corollary 2^{A}.

Equality (3.16) can be proved combinatorially. Consider a partition of n counted by $F^*(n; i, j)$. Extract the parts $\equiv 2 \pmod 4$ and form a Ferrers graph π_1 with them. Construct another Ferrers graph π_2 with the parts $\equiv 1 \pmod 4$ as follows: If the j parts $\equiv 1 \pmod 4$ are $4m_1+1$, $4m_2+1, \ldots,$ $4m_j+1$, with $m_1 > m_2 > \cdots > m_j$, then the first column of π_2 has m_1+1 nodes; this is followed by three columns of m_1 nodes each. The fifth column of π_2 has m_2+1 nodes, followed by three columns of m_2 nodes, and so on.

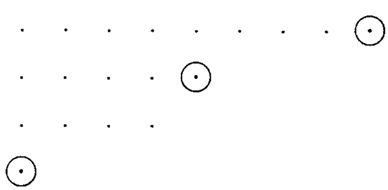
Consider the Ferrers graph π_3 obtained by adding the numbers of nodes in corresponding rows of π_1 and π_2. Note that j of the parts $\equiv 2 \pmod 4$ in π_1 become j distinct parts $\equiv 3 \pmod 4$ in π_3, in view of the extra node at the bottom of the first column, the fifth column, etc. The remaining parts of π_3 are $\equiv 2 \pmod 4$, and there are $i+j-j=i$ of them. Thus π_3 is a partition counted by $B_2^*(n; i, j)$, and this construction is reversible. This procedure works even when $m_j = 0$, that is when the smallest part $\equiv 1 \pmod 4$ is 1. In this case π_2 has an extra node at the far right of its first row. We illustrate the procedure for the partition

$$13+10+10+6+5+2+2+2+1$$

counted by $F^*(51; 3, 3)$. The extra nodes in π_2 are circled:

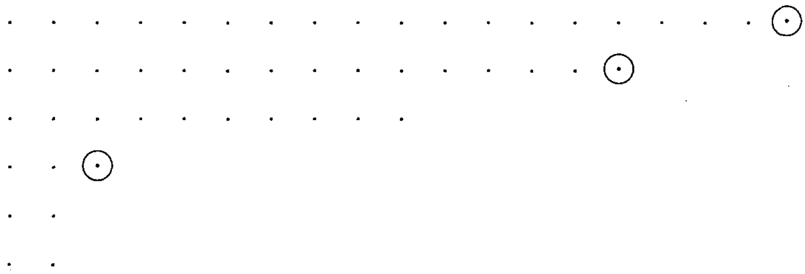
π_1

π_2



$$m_1 = 3,\ m_2 = 1,\ m_3 = 0$$

$$\pi_3\colon 19 + 15 + 10 + 3 + 2 + 2$$



All the constructions described above work more generally under the substitutions $q \mapsto q^{2m}$, $t \mapsto tq^m$, and $a \mapsto -bt^{-1}q^{\beta-m}$ in (3.12). These give

$$\sum_{k \geq 0} \frac{(-bt^{-1}q^{\beta-m};\, q^{2m})_k\, t^k q^{mk}}{(q^{2m};\, q^{2m})_k} = \prod_{r=0}^{\infty} \frac{(1 + bq^{2mr+\beta})}{(1 - tq^{(2r+1)m})}. \tag{3.17}$$

With modulus $2m$ and $\beta > m$, let

$F^*(n; i, j) =$ the number of partitions of n into j distinct $(\beta - m)$-parts each $< 2m(i+j) + (\beta - m)$ and $i+j$ m-parts

and

$B_2^*(n; i, j) =$ the number of partitions of n into j distinct β-parts and i m-parts.

Then (3.17) implies that (3.16) holds with modulus $2m$, and this is the exact analogue of equality (3.11) in Theorem A with modulus m. The functions B^* in (2.7) and B_2^* are related by

$$B^*(n; j) = \sum_i B^*(n; i, j) = \sum_i B_2^*(n; i, j), \tag{3.18}$$

because the right side of (2.8) with $a = 1$ equals the right side of (3.17) with $t = 1$. Even though (2.7) holds in the special case $\alpha = m$, the modulus of all three functions A, B, and C in Theorem A is m, whereas F^*, B^* and B_2^* all have modulus $2m$.

(v) If we set $b = 1$ in (3.8), we get partition functions

$$A(n) = \sum_k A(n; k) \qquad B^*(n) = \sum_k B^*(n; k) \qquad C(n) = \sum_k C(n; k).$$

In this case the product in (3.10) can be written as

$$\prod_{m=1}^{\infty} (1+q^{4m-1})(1+q^{4m-2})(1+q^{4m}) = \prod_{j \equiv 2,3,7 \pmod 8} (1-q^j)^{-1}. \tag{3.19}$$

The right side of (3.19) is the generating function of $D(n)$, the number of partitions of n into parts $\equiv 2, 3, 7 \pmod 8$. Thus we have

COROLLARY 3^A. *The partition functions $A(n)$, $B^*(n)$, $C(n)$, and $D(n)$ are all equal.*

When $b=1$ the product in (3.9) can be rewritten as

$$\prod_{m=1}^{\infty} (1+q^{4m-3})(1+q^{4m-2})(1+q^{4m}) = \prod_{j \equiv 1,5,6 \pmod 8} (1-q^j)^{-1}. \tag{3.20}$$

This leads to the continued fraction identity

$$1+\cfrac{q}{1+q^2+\cfrac{q^3}{1+q^4+\cfrac{q^5}{\cdots}}} = \frac{\prod_{j \equiv 2,3,7 \pmod 8} (1-q^j)}{\prod_{j \equiv 1,5,6 \pmod 8} (1-q^j)}. \tag{3.21}$$

Göllnitz [8] states the equality $B^*(n)=C(n)=D(n)$, but (3.21) seems to have escaped attention.

There is a continued fraction identity due to Gordon [9] and Göllnitz [8] which looks very similar to (3.21), namely

$$1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\cfrac{q^6}{\cdots}}} = \frac{\prod_{j \equiv 3,4,5 \pmod 8} (1-q^j)}{\prod_{j \equiv 1,4,7 \pmod 8} (1-q^j)}. \tag{3.22}$$

The numerator of (3.22) is

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(-q;q^2)_k}{(q^2;q^2)_k} = \prod_{j \equiv 1,4,7 \pmod 8} (1-q^j)^{-1}, \tag{3.23}$$

and the denominator is

$$\sum_{k=0}^{\infty} \frac{q^{k^2+2k}(-q;q^2)_k}{(q^2;q^2)_k} = \prod_{j \equiv 3,4,5 \pmod 8} (1-q^j)^{-1}. \tag{3.24}$$

The right sides of (3.23) and (3.24) have obvious partition interpretations. The series in (3.23) is the generating function of partitions with minimal difference 2 and no consecutive even parts. The series in (3.24) has a similar interpretation with the additional restriction that all parts are $\geqslant 3$.

Although these series can be obtained from (2.1) and (2.2) by choosing a, b, and q suitably, it is to be noted that the product representations in (3.23) and (3.24) do not follow directly by the methods of this paper. We also point out that identities (3.22)–(3.24) do not hold with an additional parameter b. Thus the Göllnitz–Gordon partition theorems correspond only to Corollary 3^{A} and not to the stronger Corollary 2^{A}.

(vi) We conclude this section with another proof of a result of Sylvester [14]. Lebesgue's identity (1.2) is equivalent to (see (2.2) of Lemma 1)

$$R^{\mathrm{D}}(1, b) = \prod_{m \geqslant 1} (1 + bq^{2m})(1 + q^m)$$

for which a combinatorial proof was given at the beginning of this section by interpreting both sides as generating functions of certain bipartitions. We now make a variation of this proof to establish Sylvester's theorem, which is equivalent to the identity

$$R^{\mathrm{N}}(1, b) = \prod_{m \geqslant 1} \frac{(1 + bq^{2m-1})}{(1 - q^{2m-1})},$$

by interpreting both sides as generating functions of certain ordinary partitions.

SYLVESTER'S THEOREM. *Let $p_k(n; \mathcal{O})$ be the number of partitions of n into k different odd parts.*

Let $p_k(n; \mathcal{D})$ be the number of partitions of n into distinct parts which can be grouped into k (maximal) blocks of consecutive integers.

Then $p_k(n; \mathcal{O}) = p_k(n; \mathcal{D})$.

Proof. First replace b by bq^{-1} in (1.2) to get

$$\sum_{k \geqslant 0} \frac{q^{k(k+1)/2}(-b)_k}{(q)_k} = \prod_{m=1}^{\infty} \frac{(1 + bq^{2m-1})}{(1 - q^{2m-1})}.$$

Note that

$$\prod_{m=1}^{\infty} \frac{(1 + bq^{2m-1})}{(1 - q^{2m-1})} = \prod_{m=1}^{\infty} \left(1 + \frac{(1+b)\, q^{2m-1}}{1 - q^{2m-1}}\right)$$
$$= \sum_{n,k} (1+b)^k \, p_k(n; \mathcal{O})\, q^n. \qquad (3.25)$$

On the other hand,

$$\sum_{k \geqslant 0} \frac{q^{k(k+1)/2}(-b)_k}{(q)_k} = \sum_{n,k} V^*(n;k)\, b^k q^n. \tag{3.26}$$

Here $V^*(n;k)$ is the number of bipartitions of n of the form

$$\rho = (\rho_1; \rho_2) = (a_1, ..., a_s; b_1, ..., b_k),$$

where the parts satisfy

$$\left.\begin{array}{c} a_1 > a_2 > \cdots > a_s \geqslant 1 \\ b_1 > b_2 > \cdots > b_k \geqslant 0 \\ s \geqslant b_1 + 1. \end{array}\right\} \tag{3.27}$$

Note the slight difference between (3.27) and (3.1).

As before, if the conjugate ρ_2^* is embedded in ρ_1, it produces a partition ρ_3 whose parts $c_1, ..., c_s$ contain at least $k-1$ gaps $\geqslant 2$ (because $b_k \geqslant 0$). This is equivalent to saying that ρ_3 has at least k blocks of consecutive integers. Conversely, given a partition counted by $p_l(n; \mathscr{D})$, we can extract bipartitions counted by $V^*(n;k)$ out of this in $\binom{l}{k}$ ways. Thus in analogy with (3.2) we have

$$V^*(n;k) = \sum_{l \geqslant k} \binom{l}{k} p_k(n; \mathscr{D}),$$

which is in spirit of (3.3) can be written as

$$\sum_k b^k V^*(n;k) = \sum_k (1+b)^k\, p_n(n; \mathscr{D}),$$

or equivalently

$$\sum_{n,k} b^k V^*(n;k)\, q^n = \sum_{n,k} (1+b)^k\, p_k(n; \mathscr{D})\, q^n. \tag{3.28}$$

Sylvester's theorem follows from (3.25) and (3.28) by comparing the coefficients of $(1+b)^k q^n$.

This proof of Sylvester's theorem is related to the one in Andrews [3], due to Ramamani and Venkatachaliengar, but our proof provides a connection with Theorem 1 and Ramanujan's fraction $R(a, b)$.

4. Colored Coins in a Fountain

In a recent paper [10], Odlyzko and Wilf considered the problem of enumerating $(n;k)$-fountains composed of n coins, with k coins at the base. Here by a fountain is meant an arrangement such that each coin in a higher row touches exactly two coins in the next lower row. Let $\phi(n;k)$ be the number of such fountains, and $\phi(n)=\sum_k \phi(n;k)$. They proved that

$$1+\sum_{n,k\geqslant 1}\phi(n;k)\,q^n b^k=\cfrac{1}{1-\cfrac{bq}{1-\cfrac{bq^2}{1-\cfrac{bq^3}{\cdots}}}}. \tag{4.1}$$

Here we show that the coefficients $r(n;i,j)$ in the expansion

$$\frac{1}{R(-a,-b)}=\cfrac{1}{1-\cfrac{bq}{1-aq-\cfrac{bq^2}{\cdots}}}=1+\sum_{n,i+j\geqslant 1} r(n;i,j)\,a^i b^j q^n \tag{4.2}$$

enumerate fountains of n coins colored in a manner to be specified later. This results reduces to the Odlyzko–Wilf theorem (4.1) in the special case $a=0$.

Consider the following ordered sequence of "colored integers":

$$1\prec\bar{1}\prec 2\prec\bar{2}\prec 3\prec\cdots. \tag{4.3}$$

We think of the unbarred integers as blue and the barred ones as red. For any member m of this sequence, let m' denote its successor and $n''=(n')'$.

THEOREM 4. *$r(n;i,j)$ is the number of representations of n in the form*

$$n=n_1+n_2+\cdots+n_{i+j} \tag{4.4}$$

with i barred parts and j unbarred parts satisfying

$$\left.\begin{aligned} n_1&=1\\ n_l\ \textit{barred}&\Rightarrow n_{l+1}\leqslant n_l'\\ n_i\ \textit{unbarred}&\Rightarrow n_{l+1}\leqslant n_l''.\end{aligned}\right\} \tag{4.5}$$

Proof. Let $f(n;i,j)$ be the number of representations of n given by (4.4) and (4.5); put $f(0,i,j)=\delta_{i,0}\,\delta_{j,0}$ to include the "empty" representation of

$n=0$. Let $g(n; i, j)$ be the number of "primitive" representations, namely those for which $n_l \neq 1$ for $l > 1$. Let

$$F(a, b) = 1 + \sum_{n, i+j \geqslant 1} f(n; i, j)\, a^i b^j q^n$$

and (4.6)

$$G(a, b) = \sum_{n, i+j \geqslant 1} g(n; i, j)\, a^i b^j q^n.$$

Then F and G are connected by the equation

$$F = 1 + FG. \tag{4.7}$$

To obtain (4.7), note that every non-empty representation (4.4) can be uniquely decomposed as a primitive representation of an integer $m \leqslant n$ followed by a representation of $n -.m$. This explains the term FG in (4.7); the term 1 in (4.7) counts the empty representation of $n=0$. We can rewrite (4.7) in the form

$$F = \frac{1}{1-G}. \tag{4.8}$$

Given a primitive representation ρ of $n > 1$ counted by $g(n; i, j)$, there are two cases:

Case 1. $n_2 = \bar{1}$. Here by deleting the $\bar{1}$ we obtain a primitive representation counted by $g(n-1; i-1, j)$.

Case 2. $n_2 = 2$. Consider the maximal block of parts $n_2 + n_3 + \cdots + n_k$ not containing a $\bar{1}$ or a 2 after n_2. Write $\rho = \rho_1 + \rho_2$, where

$$\rho_1 = 1 + n_{k+1} + \cdots, \qquad \rho_2 = n_2 + \cdots + n_k.$$

Subtract 1 from each part of ρ_2 (retaining colors) to get a primitive representation. Combining Cases 1 and 2, we obtain the relation

$$g(n; i, j) = g(n-1); i-1, j) + \sum_{n' \leqslant n, i' \leqslant i, j' \leqslant j} g(n-n'; i-i', j-j')\, g(n'-i'-j'; i', j')$$

for $n > 1$. In terms of generating functions this gives

$$G(a, b) = aqG(a, b) + G(a, b)\, G(aq, bq) + bq. \tag{4.9}$$

The term bq in (4.9) counts the unique primitive representation of $n=1$.

We can rewrite (4.9) in the form

$$G(a, b)=\frac{bq}{1-aq-G(aq, bq)}. \tag{4.10}$$

From (4.2), (4.8) and iteration of (4.10) it follows that $f(n; i, j)=r(n; i, j)$, proving Theorem 4.

Remarks. (i) Given an $(n; k)$-fountain, consider the number of coins along each diagonal of negative slope. This generates a representation of n in the form

$$n=n_1+n_2+\cdots+n_k, \qquad \text{with } n_1=1 \quad \text{and} \quad n_{l+1}\leqslant n_l+1. \tag{4.11}$$

Conversely, every such representation gives rise to an $(n; k)$-fountain. If instead of uncolored integers we consider colored integers ordered as in (4.3), then (4.5) implies the following: $r(n; i, j)=f(n; i, j)$ is the number of $(n; 1+j)$-fountains with i red diagonals and j blue diagonals such that

(a) the first diagonal (i.e., the leftmost coin) is blue and

(b) a red diagonal cannot be followed by a longer red diagonal.

(ii) When $a=0$ in (4.2), all coins are colored blue. In this case we have the interpretation that $\phi(n; k)$ is the number of representative satisfying (4.11).

(iii) Consider the dilatation $q\mapsto q^2$ and the translation $b\mapsto bq^{-1}$ in (4.2). This corresponds to the continued fraction (3.8), which has many interesting properties (Section 5). In this case we have the odd positive integers colored blue and the even ones colored red. Let

$$\cfrac{1}{1-\cfrac{bq}{1-aq^2-\cfrac{bq^3}{1-aq^4-\cfrac{bq^5}{\cdots}}}}=1+\sum r_1^{(2)}(n; i, j)\, a^i b^j q^n. \tag{4.12}$$

Then the coefficients $r_1^{(2)}(n; i, j)$ count the representations of n in the form

$$n=n_1+n_2+\cdots+n_{i+j}$$

with i even (red) parts and j odd (blue) parts such that $n_1=1$ and

$$\left.\begin{aligned} n_l\,\text{odd} &\Rightarrow n_{l+1}\leqslant n_l+2\\ n_l\,\text{even} &\Rightarrow n_{l+1}\leqslant n_l+1.\end{aligned}\right\} \tag{4.13}$$

Instead of (4.12) we may consider in analogy with the Göllnitz–Gordon fraction (3.22) the expansion

$$\cfrac{1}{1-aq-\cfrac{bq^2}{1-aq^3-\cfrac{bq^4}{\cdots}}}=1+\sum r_2^{(2)}(n;i,j)\,a^ib^jq^n.$$

Here $r_2^{(2)}(n;i,j)$ is the number of representations $n=n_1+n_2+\cdots+n_{i+j}$ with i odd and j even parts such that $n_1=1$ or 2 and

$$\left.\begin{aligned} n_l\text{ even} &\Rightarrow n_{l+1}\leqslant n_l+2\\ n_l\text{ odd} &\Rightarrow n_{l+1}\leqslant n_l+1.\end{aligned}\right\}\tag{4.14}$$

Condition (4.14) is dual to (4.13).

5. Concluding Remarks

If we put $b=1$ and let $q\to 1^-$ in (3.8), then we get from (3.13)

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cdots}}}}=\prod_{m=0}^{\infty}\frac{(8m+2)(8m+3)(8m+7)}{(8m+1)(8m+5)(8m+6)}.\tag{5.1}$$

It is possible to give a "direct" proof of (5.1) using properties of the Gamma function. For this, note that the product in (5.1) is

$$\prod_{m=0}^{\infty}\frac{(m+\frac{2}{8})(m+\frac{3}{8})(m+\frac{7}{8})}{(m+\frac{1}{8})(m+\frac{5}{8})(m+\frac{6}{8})}=\frac{\Gamma(\frac{1}{8})\,\Gamma(\frac{5}{8})\,\Gamma(\frac{6}{8})}{\Gamma(\frac{2}{8})\,\Gamma(\frac{3}{8})\,\Gamma(\frac{7}{8})}\tag{5.2}$$

because of Gauss' formula

$$\lim_{n\to\infty}\frac{z(z+1)(z+2)\cdots(z+n)}{n!\,n^z}=\frac{1}{\Gamma(z)}$$

and also because

$$1+5+6=2+3+7.\tag{5.3}$$

We now use the duplication formula of Gauss, which is

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2z)}{2^{2z-1}}.$$

Observe that

$$\tfrac{5}{8}=\tfrac{1}{8}+\tfrac{1}{2} \qquad \text{and} \qquad \tfrac{7}{8}=\tfrac{3}{8}+\tfrac{1}{2}$$

and so, by the duplication formula, the product of Gamma functions in (5.2) can be rewritten as

$$\frac{\sqrt{\pi}\,\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})\,2^{3/4-1}}{2^{1/4-1}\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})\sqrt{\pi}}=\sqrt{2}.$$

This proves (5.1).

Recently it was noticed by Andrews *et al.* [6] that the convergents to the continued fraction (3.8) remain invariant if q is replaced by q^{-1}. That is,

$$1+\cfrac{bq^{-1}}{1+q^{-2}+\cfrac{bq^{-3}}{1+q^{-4}+\ddots\cfrac{bq^{1-2n}}{1+q^{-2n}}}}$$

$$=1+\cfrac{bq}{1+q^{2}+\cfrac{bq^{3}}{1+q^{4}+\ddots\cfrac{bq^{2n-1}}{1+q^{2n}}}}$$

In [6] this invariance is stated only in the case $b=1$ for the purpose of proving that the continued fraction converges for $|q|>1$. On the other hand this invariance does not hold for the product

$$\prod_{m=0}^{\infty}\frac{(1+bq^{4m+1})}{(1+bq^{4m+3})}$$

even when $b=1$. However, when $b=1$, the above product can be rewritten as

$$\prod_{m=0}^{\infty}\frac{(1-q^{8m+2})(1-q^{8m+3})(1-q^{8m+7})}{(1-q^{8m+1})(1-q^{8m+5})(1-q^{8m+6})},$$

and we note that this remains invariant when q is replaced by q^{-1} because of (5.3).

It is quite common for products to remain invariant under the transformation $q \mapsto q^{-1}$ because all that is required is to have a modulus m and two sets of residues S_1 and S_2 mod m such that

$$\sum_{r_1 \in S_1} r_1 = \sum_{r_2 \in S_2} r_2.$$

For then

$$\prod (1-q^{\mu}) \Big/ \prod (1-q^{\nu})$$

has the invariance property if μ and ν run through all positive integers in the residue sets S_1 and S_2, respectively. Thus the product in the Gordon–Göllnitz continued fraction (3.14) as well as the Rogers–Ramanujan product remain invariant under $q \to q^{-1}$. But neither of the corresponding continued fractions has this feature. So this invariance is one of the special features of the continued fraction (3.8).

Ramanujan [12] obtained among others the following continued fraction with three parameters a, b, q possessing a product representation:

$$1 - ab + \cfrac{(a-bq)(b-aq)}{(1-ab)(1+q^2) + \cfrac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4) + \cdots}}$$
$$= \prod_{m=1}^{\infty} \frac{(1-a^2q^{4m-3})(1-b^2q^{4m-3})}{(1-a^2q^{4m-1})(1-b^2q^{4m-1})}. \tag{5.4}$$

This has been proved only recently [1]. If we put $a=0$ and replace b^2 by $-b$ in (5.4), we get (3.8). It would be worthwile to study the combinatorial properties of the coefficients in the power series expansion of this fraction.

ACKNOWLEDGMENT

We thank the referee for pointing out some mis-statements in an earlier version of this paper.

REFERENCES

1. C. ADIGA, B. C. BERNDT, S. BHARGAVA, AND G. N. WATSON, "Chapter 16 of Ramanujan's Second Notebook: Theta Functions and q-Series," Memoirs of the American Mathematical Society, Vol. 315, Amer. Math. Soc., Providence, RI, 1985.
2. G. E. ANDREWS, On generalizations of Euler's partition theorem, *Michigan Math. J.* **13** (1966), 491–498.
3. G. E. ANDREWS, *in* "Encyclopedia of Mathematics," Vol. 2, Addison–Wesley, Reading, MA, 1976.

4. G. E. ANDREWS, An introduction to Ramanujan's "lost" notebook, *Amer. Math. Monthly* **86** (1979), 89–108.
5. G. E. ANDREWS, Ramanujan's "Lost" Notebbook. III. The Rogers–Ramanujan continued fraction, *Adv. Math.* **41** (1981), 186–208.
6. G. E. ANDREWS, B. C. BERNDT, L. JACOBSEN, AND R. L. LAMPHERE, "The Continued Fractions Found in the Unorganized Portions of Ramanujan's Notebooks," Memoirs of the American Mathematical Society, Vol. 417, Amer. Math. Soc., Providence, RI, 1992.
7. D. M. BRESSOUD, On a partition theorem of Göllnitz, *J. Reine Angew. Math.* **305** (1979), 215–217.
8. H. GÖLLNITZ, Partitionen mit Differenzenbedingungen, *J. Reine Angew. Math.* **225** (1967), 154–190.
9. B. GORDON, Some continued fractions of the Rogers–Ramanujan type, *Duke Math. J.* **32** (1965), 741–748.
10. A. M. ODLYZKO AND H. S. WILF, *n* coins in a fountain, *Amer. Math. Monthly* **95** (1988), 840–843.
11. S. RAMANUJAN, "Collected Papers," Cambridge Univ. Press, London/New York, 1927.
12. S. RAMANUJAN, "Notebooks (Two Volumes)," Tata Institute, Bombay, 1957.
13. S. RAMANUJAN, "The Lost Notebook, and Other Unpublished Papers," Narosa, New Delhi, 1988.
14. J. J. SYLVESTER, A constructive theory of partitions arranged in three Acts, and Interact, and an Exodion, *Amer. J. Math.* **5** (1882), 251–330.